The shallow water equations: explicit solutions and superposition principle

1983 J. Phys. A: Math. Gen. 16 2133

(http://iopscience.iop.org/0305-4470/16/10/009)

J. Phys. A: Math. Gen. 16 (1983) 2133–2136. Printed in Great Britain

The shallow water equations: explicit solutions and superposition principle

Yilmaz Akyildiz†
Department of Mathematical Sciences, University of Petroleum and Minerals, Dhahran, Saudi Arabia

Received 10 November 1982

Abstract. By using the hodograph method, we find almost all the solutions of the classical shallow water equations. We also construct an auto-Bäcklund transformation (superposition principle) on the set of these hodograph solutions and show that this transformation is canonical relative to a symplectic form introduced by Manin.

In their paper, Cavalcante and McKean (1982) wish to know an explicit solution to the following classical shallow water equations on a horizontal bottom:

$$u_t + uu_x + h_x = 0, \qquad h_t + hu_x + uh_x = 0, \tag{1a, b}$$

where $-\infty < x < \infty$ is the horizontal coordinate, t is the time, $u = u(t, x)$ is the horizontal component of the velocity at the point x at time t, and $h = h(t, x) > 0$ is the height of the free surface above the point x at time t. In our paper (Akyildiz 1982), besides other things, we have constructed explicit solutions to this system of equations. It was interesting to see that these solutions and the conservation laws of this system could be systematically obtained from a single wave equation. In this article, after outlining the construction of these solutions, we shall show that the solution space of the wave equation (from which we obtain solutions and conservation laws for the system (1)) is *isotropic* relative to a symplectic form introduced by Manin (1978), hence simultaneously proving that the conserved quantities are in involution (which was known to Manin) and that the accompanying Bäcklund transformation is symplectic. It is an important question whether the isotropic space of the solutions of the above-mentioned wave equation is *Lagrangian*. This is the problem of complete integrability of (1), which is an infinite-dimensional Hamiltonian system. See §§ 7 and 9 in Cavalcante and McKean (1982) on this important matter.

The system (1) is a pair of quasi-linear partial differential equations with no explicit (t, x) dependence. Hence, for any region where the Jacobian $j = u_x h_t - u_t h_x$ is non-zero, (1) can be transformed into an equivalent linear system by interchanging the roles of dependent and independent variables, (u, h) and (t, x), respectively. This is a so-called *hodograph transformation*. Since (1) is homogeneous, from $u_x = jt_h$, $u_t = -jx_h$, $h_x = -jt_u$, $h_t = jx_u$, we see that the highly nonlinear factor j cancels through in (1), and we arrive

† On leave from the Middle East Technical University, Ankara, Turkey.

at the following linear differential equations:

$$x_u = ut_u - ht_h, \qquad x_h = -t_u + ut_h, \qquad \text{or} \qquad \nabla x = \begin{bmatrix} u & -h \\ -1 & u \end{bmatrix} \nabla t, \tag{2a, b}$$

where ∇ is the gradient operator $(\partial/\partial u, \partial/\partial h)$ on the (u, h) plane. By eliminating x in (2) we obtain the linear equation

$$t_{uu} = 2t_h + ht_{hh},$$

whose solutions can easily be found in standard tables.

Since the application of the hodograph transformation depends on the assumption that $j \neq 0$, solutions for which $j = 0$ cannot be obtained by this method. Such solutions are called *simple waves*, and they form a set of measure zero in the set of all solutions of the system (1). Simple waves are important tools for the solution of flow problems; for instance, wave breaking occurs when $j = 0$ due to the multivaluedness, i.e. shocks. As an example, the solution

$$u = 2x/3t, \qquad h = (x/3t)^2,$$

which is found in Nutku (1981) by a scale-invariance argument, represents a simple wave. So, we could not possibly obtain it by the hodograph method.

The system of equations (2) can also be written in the equivalent form

$$(x - ut)_u = -(ht)_h, \qquad (x - ut)_h = -t_u. \tag{3a, b}$$

These, in return, suggest the existence of potentials $\Psi(u, h)$ and $\Phi(u, h)$ satisfying

$$\Psi_u = -ht, \qquad \Psi_h = x - ut, \tag{4a}$$

$$\Phi_u = x - ut, \qquad \Phi_h = -t. \tag{4b}$$

Solving (4) for x and t we obtain

$$x = \Psi_h - (u/h)\Psi_u, \qquad t = -(1/h)\Psi_u, \tag{5a}$$

$$x = \Phi_u - u\Phi_h, \qquad t = -\Phi_h. \tag{5b}$$

Combining (5a) with (3b) gives the following *wave equation*:

$$\Psi_{uu} = h\Psi_{hh}. \tag{6}$$

Similarly, (5b) together with (3a) gives

$$\Phi_{uu} = (h\Phi_h)_h. \tag{7}$$

Thus, the potentials Ψ and Φ satisfy linear equations whose solutions can be obtained by standard methods. From any one of these potentials, via (5), we can easily construct hodograph solutions of our original system (1). For example, let us construct some polynomial solutions to (6) in the form

$$\Psi = \sum_{i=0}^{n} p_i(u)h^i,$$

and then find the corresponding solutions of (1). We must satisfy the relations

$$p_n'' = 0, \qquad p_{m-1}'' = m(m-1)p_m(u), \qquad m = 1, 2, \ldots, n.$$

Below we list the first few of these polynomials and solutions.

Ψ	x	t	u	h
u	u/h	$1/h$	x/t	$1/t$
h	1	0	—	—
uh	0	1	—	—
$\frac{1}{2}(u^2h+h^2)$	$h-u^2/2$	$-u$	$-t$	$x+t^2/2$
$\frac{1}{3}u^3h+uh^2$	$\frac{2}{3}u^3-uh$	u^2+h	Implicit solution	

In Akyildiz (1982), we have constructed conserved quantities, $\int \Psi\, dx$ where Ψ is a solution of (6), for the system (1)†. Thus, the solution space of (6) serves as a *moduli space* for the hodograph solutions and also for the conservation laws of the system (1). By using this correspondence and the linear nature of (6) we can construct an auto-Bäcklund transformation on the hodograph solutions. This is, of course, nothing but a *superposition principle* for the *nonlinear* system (1): given two solutions of (1), add the corresponding solutions of (6), and then construct the solution of (1) corresponding to this sum.

In 1978, Manin introduced the following *Hamiltonian formalism* for the system (1):

$$\begin{bmatrix} u \\ h \end{bmatrix}_t = J\nabla H, \qquad J = -\begin{bmatrix} 0 & D \\ D & 0 \end{bmatrix}, \qquad H = \tfrac{1}{2}(u^2h+h^2),$$

where J is the Hamiltonian operator, H the Hamiltonian, ∇ the gradient operator in (u, h)-space, and D the differentiation with respect to x. The associated *Poisson bracket* is defined to be

$$[A, B] = \int_{-\infty}^{\infty} \nabla A J \nabla B \, dx \tag{8}$$

for two functions A and B of the variables u and h.

Finally, we show that this Poisson bracket vanishes on the solution space of the wave equation (6), which parametrises both the hodograph solutions and conservation laws of the system (1). Let A and B be two solutions of (6). The integrand in (8) is

$$(A_uDB_h + A_hDB_u)\, dx = A_u\, dB_h + A_h\, dB_u.$$

Since

$$\begin{aligned}
d(A_u\, dB_h + A_h\, dB_u) & \\
&= dA_u \wedge dB_h + dA_h \wedge dB_u \\
&= (A_{uu}\, du + A_{uh}\, dh) \wedge (B_{hu}\, du + B_{hh}\, dh) \\
&\quad + (A_{hu}\, du + A_{hh}\, dh) \wedge (B_{uu}\, du + B_{uh}\, dh) \\
&= (A_{uu}B_{hh} - A_{hh}B_{uu})\, du \wedge dh \\
&= (hA_{hh}B_{hh} - A_{hh}hB_{hh})\, du \wedge dh \qquad \text{by (6)} \\
&= 0,
\end{aligned}$$

† The transformation equation between this article and Akyildiz (1982) is $h = c^2$.

the integrand in (8) is closed and is, therefore, exact on simply connected regions; that is,

$$[A, B] = \int_{-\infty}^{\infty} \mathrm{d}C = 0,$$

for a function C of u and h. (We assume that u and h vanish at spatial infinities.) Thus, we have simultaneously proved that the auto-Bäcklund transformation above is symplectic and that conserved quantities constructed in Akyildiz (1982) are in involution.

Acknowledgments

We would like to thank M F Atiyah for the most useful conversations we had with him during the First International Conference on Mathematics in the Arabian Gulf Area, Riyadh, October 1982. We would also like to acknowledge the University of Petroleum and Minerals where this research has been done.

Note added in proof. Recently, professor Yu I Manin informed us that his paper with B A Kuperschmidt, 1977 *Funct. Anal.* **11** 31 (in Russian), also contains explicit solutions to the system of equations (1). See also the book *Water Waves* by J J Stoker (1957), p 337. We would like to thank Professor Manin for his cooperation.

References

Akyildiz Y 1982 *J. Math. Phys.* **9** 1723
Cavalcante J and McKean H P 1982 *Physica* **4D** 253
Manin Yu I 1978 *Sov. Prob. Mat.* **11** 5
Nutku Y 1981 unpublished